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Letter to the Editor

A note on compact normal operators

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Abstract

In this note, we present three simple necessary and sufficient conditions for a linear compact operator on a Hilbert space H to be normal.

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It is well known that linear compact operators on a Hilbert space form one of the most important class of bounded linear operators and have many more excellent properties, see [1]. Thus, it is very important to consider some conditions under which a linear compact operator is normal. Recently, in [3] Sadkane presented three simple necessary and sufficient conditions under which a n-by-n complex matrix is normal. Since n-by-n complex matrices can be considered as the class of linear compact operators on Hilbert space $\mathbb{C}^n$, it is natural to ask whether those results in [3] hold for all linear compact operators on a separable complex Hilbert space. In this note, we give an affirmative answer to this question, and our proofs are different from [3].

Let $\mathbb{N}, \mathbb{C}$ be the sets of all positive integers and all complex numbers, respectively. And denote by $\aleph_0$ the cardinal of $\mathbb{N}$. Let H be a separable complex Hilbert space and I be the identity operator on H. In the note, we always assume that $A : H \to H$ is a linear compact operator on H, and let $\mathbf{r}(A), N(A)$ be the spectral radius and the null space of A, respectively. We denote by $\lambda_1(A), \lambda_2(A), \lambda_3(A), \ldots$ all the eigenvalues of A. We assume that the eigenvalues are ordered such that

$$|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant |\lambda_3(A)| \geqslant \cdots, \tag{1}$$

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where each eigenvalue is repeated as many times as the value of its multiplicity. Let $\lambda_1(A^*A), \lambda_2(A^*A), \ldots$ be the sequence of eigenvalues of A^*A such that

$$\lambda_1(A^*A) \geqslant \lambda_2(A^*A) \geqslant \lambda_3(A^*A) \geqslant \cdots, \tag{2}$$

where each eigenvalue is repeated as many times as its multiplicity. By definition, for $j = 1, 2, \ldots$ the jth singular value of A is the number $s_j(A) \stackrel{\text{def}}{=} (\lambda_j(A^*A))^{1/2}$, see for example [2, p. 96]. Without loss of generality, we assume that there exists a sequence $\{m_i\}_{i=1}^{v} \subseteq \mathbb{N}$ with $m_l < m_k$, *for* $l < k$ such that $\lambda_{m_l}(A) \neq \lambda_{m_k}(A)$, *for* $l \neq k$ and $\lambda_j(A) = \lambda_{m_i}(A)$ *for* $j \in \{m_{i-1}+1, m_{i-1}+2, \ldots, m_i\}$, where $i = 1, 2, \ldots, v$ and $m_0 = 0$. Clearly v may be a finite number or $\aleph_0$. With these symbols, we have

Lemma 1. *Suppose A be a compact operator on H. If $\lambda_j(A) \neq 0$ and $s_j(A) = |\lambda_j(A)|$, for all $j \in \mathbb{N}$, then, for each $s \in \{1, \ldots, v\}$ with $s < +\infty$, there exists an orthonormal set $\{x_1, x_2, \ldots, x_{m_s}\}$ of H such that*

$$A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s$$

and $\{\lambda_{m_1}(A), \ldots, \lambda_{m_s}(A)\} \subseteq \mathbb{C} \setminus \sigma(A_s)$, where A_s is a compact operator on $\bigcap_{i=1}^{s} N(A - \lambda_{m_i}(A)I)^{\perp}$.

Proof. First, we will prove that $N(A - \lambda_{m_1}(A)I)$ is a reduced subspace for A. To do this, it is enough to show that $N(A - \lambda_{m_1}(A)I)$ is invariant for A^*. Note that $\lambda_1(A) = \cdots = \lambda_{m_1}(A)$. For $j = 1$, we have

$$|\lambda_1(A)| = s_1(A) = \max_{0 \neq x \in H} \frac{\|Ax\|}{\|x\|} = \|A\|.$$

Then for each $x \in N(A - \lambda_{m_1}(A)I)$ with $\|x\| = 1$, we have

$$\|A\| = |\lambda_1(A)| = |\langle Ax, x\rangle| = |\langle x, A^*x\rangle| \leqslant \|A^*x\| \leqslant \|A^*\| = \|A\|.$$

So $|\langle x, A^*x\rangle| = \|A^*x\|$. By Schwarz inequality, there exists $h \in \mathbb{C}$ such that $A^*x = hx$. Furthermore, we can get $h = \overline{\lambda_{m_1}(A)}$. Thus $A^*x \in N(A - \lambda_{m_1}(A)I)$. This shows that $N(A - \lambda_{m_1}(A)I)$ is a reduced subspace for A. Therefore the restriction of A to $N(A - \lambda_{m_1}(A)I)$ is $\lambda_{m_1}(A)I_1$, where I_1 is the identity operator on $N(A - \lambda_{m_1}(A)I)$. Hence $A = \lambda_{m_1}(A)P_1 \oplus A_1$, and $\lambda_{m_1}(A) \in \mathbb{C} \setminus \sigma(A_1)$, where P_1 is the orthogonal projection on $N(A - \lambda_{m_1}(A)I)$. Moreover, we assume that $\{x_1, \ldots, x_{m_1}\}$ is an orthonormal basis of $N(A - \lambda_{m_1}(A)I)$, then $\lambda_{m_1}(A)P_1 = \sum_{j=1}^{m_1} \lambda_{m_1}(A)(x_j \otimes x_j)$, that is, $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$. Repeat the process above, we can obtain $A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s$. $\square$

The following is the main result of this paper.

Theorem 2. *Let A be a compact operator on H, then the following statements are equivalent*:

(a) *A is normal*
(b) *For each $j \in \mathbb{N}$,*

$$s_j(A^{m+n}) = \sqrt{s_j(A^{2n})s_j(A^{2m})}, \quad \forall m, n = 0, 1, 2, \ldots \, . \tag{3}$$

(c) $s_j(A) = |\lambda_j(A)|$, *for* $j = 1, 2, \ldots$.
(d) *For each* $x \in H$,

$$\|A^{n+m}x\| \leqslant \sqrt{\|A^{2n}x\| \cdot \|A^{2m}x\|}, \quad \forall n, m = 0, 1, 2, \ldots \ . \tag{4}$$

Proof. If H is finite dimensional, then A can be viewed as a matrix, the proof can be found in [3]. Next, we only consider the case H is infinite dimensional.

(a) $\Rightarrow$ (b): Suppose that A is normal, then $A = \sum_{i=1}^{v} \lambda_{m_i}(A)P_i$, where P_i is the orthogonal projection onto $N(A - \lambda_{m_i}(A)I)$. Hence $A^*A = \sum_{i=1}^{v} |\lambda_{m_i}(A)|^2 P_i$ and $(A^n)^*A^n = (A^*A)^n = \sum_{i=1}^{v} |\lambda_{m_i}(A)|^{2n} P_i$. Therefore

$$\{s_{m_i}(A)\colon\ i = 1, 2, \ldots, v\} = \{|\lambda_{m_i}(A)|\colon\ i = 1, 2, \ldots, v\}.$$

Clearly,

$$\begin{aligned}\dim(N((A^*A)^n - (s_{m_i}(A))^{2n})) &= \dim(N(A^*A - (s_{m_i}(A))^2)) \\ &= \sum_{k \in D} \dim(N(A - \lambda_k(A))),\end{aligned}$$

where $D = \{k\colon\ |\lambda_k(A)| = s_{m_i}(A)\}$. Since $(|\lambda_j(A)|)_{j=1}^{\infty}$ and $(s_j(A))_{j=1}^{\infty}$ are ordered as (1) and (2), respectively, we have

$$s_j(A^n) = (s_j(A))^n = |\lambda_j(A)|^n. \text{ Therefore}$$

$$s_j(A^{m+n}) = \sqrt{s_j(A^{2n}) s_j(A^{2m})},\ \forall_j \in \mathbb{N},\ \ \forall n, m = 0, 1, 2, \ldots \ .$$

(b) $\Rightarrow$ (c): Under condition (3), we have

$$s_j(A) = (s_j(A^2))^{1/2} = (s_j(A^4))^{1/4} = \cdots = (s_j(A^{2^k}))^{1/2^k} = \cdots \quad \forall k \in \mathbb{N}.$$

If $j = 1$, then $s_1(A^m) = \max_{0 \neq x} \|A^m x\|/\|x\| = \|A^m\|$, hence

$$s_1(A) = \lim_{k\to\infty} (s_1(A^{2^k}))^{1/2^k} = \lim_{k\to\infty} \|A^{2^k}\|^{1/2^k} = \mathbf{r}(A) = |\lambda_1(A)|.$$

Hence $s_1(A) = \cdots = s_{m_1}(A) = |\lambda_1(A)| = \cdots = |\lambda_{m_1}(A)|$. From the proof of Lemma 1, we get $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$ and $s_{m_1+1}(A) = s_1(A_1), \lambda_{m_1+1}(A) = \lambda_1(A_1)$. Hence,

$$s_1(A_1) = (s_1(A_1^2))^{1/2} = \cdots = (s_1(A_1^{2^k}))^{1/2^k} = \cdots \ \ \forall k \in \mathbb{N}$$

and

$$s_1(A_1) = \lim_{k\to\infty} (s_1(A_1^{2^k}))^{1/2^k} = \lim_{k\to\infty} \|A_1^{2^k}\|^{1/2^k} = |\lambda_1(A_1)|.$$

So $s_{m_1+1}(A) = \cdots = s_{m_2}(A) = |\lambda_{m_1+1}(A)| = \cdots = |\lambda_{m_2}(A)|$. For each $j > 1$, by induction and simple discussion, we can obtain $s_j(A) = |\lambda_j(A)|$.

(c) $\Rightarrow$ (a): By Lemma 1, for each $s \in \{1, \ldots, v\}$ with $s < +\infty$, we have

$$A = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j) \oplus A_s.$$

Let $B_s = \sum_{j=1}^{m_s} \lambda_j(A)(x_j \otimes x_j)$, then $A = B_s \oplus A_s$. If v is a finite number, then clearly $A = B_v \oplus 0$ is normal. In the case $v = \aleph_0$, let $(\alpha_{sk})_{k=1}^{\infty}$ be the sequence of distinct eigenvalues of $A_s^* A_s$, with $\alpha_{sk} \geqslant \alpha_{si}$, $k > i$ and P_{sk} be the orthogonal projection from $\bigcap_{i=1}^{s} N(A - \lambda_{m_i}(A)I)^{\perp}$ onto $N(A_s^* A_s - \alpha_{sk} I_s)$ and $P_{sk}P_{st} = P_{st}P_{sk} = 0$ *for* $k \neq t$, where I_s is the identity operator on $\bigcap_{i=1}^{s} N(A - \lambda_{m_i}(A)I)^{\perp}$. Clearly $(\alpha_{sk})_{k=1}^{\infty}$ is a subsequence of $((s_j(A))^2)_{j=1}^{\infty}$. Then $A_s^* A_s = \sum_{k=1}^{\infty} \alpha_{sk} P_{sk}$, since $A_s^* A_s$ is compact and normal. For each $x \in \bigcap_{i=1}^{s} N(A - \lambda_{m_i}(A)I)^{\perp}$, we have

$$\|(A_s^* A_s)x\|^2 = \sum_{k=1}^{\infty} \alpha_{sk}^2 \|P_{sk}x\|^2 \leqslant \sup\{\alpha_{sk}^2 \colon k \geqslant 1\} \cdot \|x\|^2.$$

Hence $\|(A_s^* A_s)\| = \sup_{\|x\|=1} \|(A_s^* A_s)x\| \leqslant \sup\{\alpha_{sk}^2 \colon k \geqslant 1\}$. Since $(s_j(A)) \to 0$ as $j \to +\infty$, we have $\sup\{\alpha_{sk}^2 : k \geqslant 1\} \to 0$ *as* $j \to \infty$. So that $\|A_s^* A_s\| \to 0$ *as* $s \to +\infty$. This implies $\|A_s\| \to 0$. Hence $\|B_s \oplus 0 - A\| = \|0 \oplus A_s\| = \|A_s\| \to 0$. Therefore A is normal, since each $B_s \oplus 0$ is normal.

(a) $\Rightarrow$ (d): If A is normal, then for each $x \in H$,

$$\begin{aligned} \|A^{n+m}x\|^2 &= \langle A^{n+m}x, A^{n+m}x\rangle \\ &= \langle (A^*A)^n x, (A^*A)^m x\rangle \\ &\leqslant \|(A^*A)^n x\| \cdot \|(A^*A)^m x\| \\ &= \|A^{2n}x\| \cdot \|A^{2m}x\|, \end{aligned}$$

since $\|(A^*A)^n x\|^2 = \langle (A^*A)^n x, (A^*A)^n x\rangle = \|A^{2n}x\|^2$. Hence, (d) holds.

(d) $\Rightarrow$ (c): condition (4) implies that

$$s_1(A^{n+m}) \equiv \max_{0 \neq x \in H} \frac{\|A^{n+m}x\|}{\|x\|} \leqslant \sqrt{s_1(A^{2n}) s_1(A^{2m})}$$

and in particular, we have

$$|\lambda_1(A^n)| \leqslant s_1(A^n) \leqslant (s_1(A^{2n}))^{1/2} \leqslant (s_1(A^{4n}))^{1/4} \leqslant \cdots \leqslant |\lambda_1(A^n)|.$$

The last inequality holds, since $\lim_{m\to\infty} (s_1(A^{nm}))^{1/m} = |\lambda_1(A^n)|$. In particular, for $n=1$, we obtain that $s_1(A) = \cdots = s_{m_1}(A) = |\lambda_1(A)| = \cdots = |\lambda_{m_1}(A)|$. From the proof of Lemma 1, we have $A = \sum_{j=1}^{m_1} \lambda_j(A)(x_j \otimes x_j) \oplus A_1$, and A_1 is a compact operator on $N(A - \lambda_{m_1}(A)I)^{\perp}$. For each $x \in N(A - \lambda_{m_1}(A)I)^{\perp}$, we have $A^{n+m}x = A_1^{n+m}x$, hence

$$\|A_1^{n+m}x\| \leqslant \sqrt{\|A_1^{2n}x\| \; \|A_1^{2m}x\|}, \quad \forall n, m = 0, 1, 2, \ldots .$$

Similarly, we get $|\lambda_1(A_1)| = s_1(A_1)$. From the proof of Lemma 1 again, we obtain

$$s_{m_1+1}(A) = \cdots = s_{m_2}(A) = s_1(A_1) = |\lambda_1(A_1)| = |\lambda_{m_1+1}(A)| = \cdots = |\lambda_{m_2}(A)|.$$

For each $j \in \mathbb{N}$, by induction and simple discussion, we get $s_j(A) = |\lambda_j(A)|$. Hence (c) holds. This completes the proof. $\square$

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